Math 2020B Midterm **Name and ID:**____________________________

This exam contains 9 pages (including this cover page) and 8 problems. The last page is the scratch paper. If any pages are missing, please inform the instructor immediately.

Directions:

- Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.
- You may *not* use your calculators, books and notes.
- You must show all your work. A stranger should be able to pick up your paper and follow your reasoning. You must use proper notation and define the events clearly. Partial credits may be given.

I do not lie, cheat or steal, or tolerate those who do.

By signing below you indicate that all your work is your own and that you have neither given nor received help from external sources.

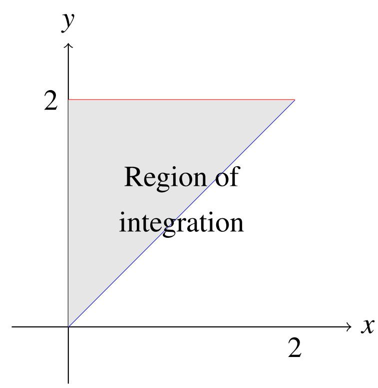
Signature: ________________________________

Problem #	Score	5	
1		6	
2		7	
3		8	
4		Total	

Problem 1: Evaluate the integral

$$I = \int_0^2 \int_x^2 2y^2 \sin(xy)\,dy\,dx.$$

Draw the area of integration. Indicate any major theorem you use.



Using Fubini's Theorem, we can rewrite I as:

$$I = \int_0^2 \int_0^y 2y^2 \sin(xy)\,dx\,dy$$

First, integrate with respect to x:

$$\int_0^y 2y^2 \sin(xy)\,dx = 2y^2 \left[-\frac{\cos(xy)}{y}\right]_0^y = 2y^2 \left(-\frac{\cos(y^2)}{y} + \frac{\cos(0)}{y}\right)$$

$$= 2y^2 \left(\frac{1-\cos(y^2)}{y}\right) = 2y(1-\cos(y^2))$$

Now integrate with respect to y:

$$I = \int_0^2 2y(1-\cos(y^2))\,dy = \int_0^2 2y\,dy - \int_0^2 2y\cos(y^2)\,dy = 4 - \sin(4).$$

Problem 2: Find the area outside the cardioid $r = 2 + 2\sin(\theta)$ but inside the circle $r = 6\sin(\theta)$.

Here is a picture:

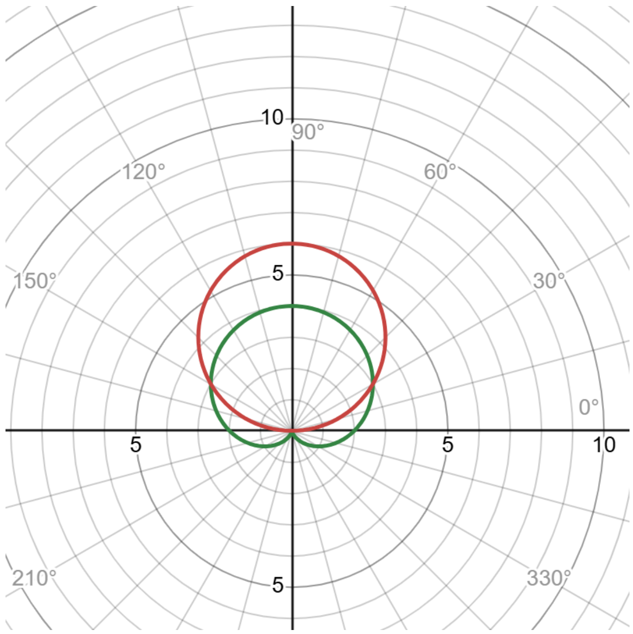


Figure 1: Enter Caption

Note that the angle should range from $\pi/6$ to $5\pi/6$. Hence you should compute

$$\begin{aligned} A &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[(6\sin(\theta))^2 - (2+2\sin(\theta))^2\right] d\theta \\ &= \frac{1}{2}\int_{\pi/6}^{5\pi/6} \left[36\sin^2(\theta) - (4+8\sin(\theta)+4\sin^2(\theta))\right] d\theta \\ &= \frac{1}{2}\left[12\theta - 8\sin(2\theta) + 8\cos(\theta)\right]_{\pi/6}^{5\pi/6} \\ &= 4\pi \end{aligned}$$

Problem 3: Let R be the region in the first quadrant of the xy-plane bounded by the curves $y = 2x$, $y = 3x$, $xy = 1$, and $xy = 9$. Use the transformation $u = y/x$ and $v = xy$ to evaluate

$$\iint_R \left(\frac{y}{x} + xy\right) dx\, dy.$$

Express x and y in terms of u and v:

$$x = \sqrt{\frac{v}{u}}, \quad y = \sqrt{uv}.$$

Compute the Jacobian determinant:

$$|J| = \frac{1}{2u}.$$

The original integral becomes:

$$\iint_{R'} (u+v) \cdot \frac{1}{2u}\, du\, dv,$$

where R' is the transformed region:

$$2 \le u \le 3, \quad 1 \le v \le 9.$$

Therefore, the answer is

$$\frac{1}{2}\int_2^3 \int_1^9 \left(1 + \frac{v}{u}\right) dv\, du = 4 + 20\ln\left(\frac{3}{2}\right).$$

Problem 4: Find the volume of the solid bounded above by $z = 4 - 4(x^2 + y^2)$ and above by $z = (x^2 + y^2)^2 - 1$.

Set the two surfaces equal to each other to find the curve of intersection:

$$4 - 4(x^2 + y^2) = (x^2 + y^2)^2 - 1.$$

We find that the curve of intersection is the circle $x^2 + y^2 = 1$ where $z = 0$.
The volume integral, in terms of polar coordinates, becomes:

$$V = \int_0^{2\pi} \int_0^1 \int_{(r^4-1)}^{(4-4r^2)} r dz dr d\theta = \frac{8\pi}{3}.$$

Problem 5: Evaluate the integral

$$\iiint_D z^3\,dx\,dy\,dz$$

over the upper half of the solid ellipsoid

$$D = \left\{(x,y,z) : \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1,\, z \ge 0\right\},$$

where $a,b,c > 0$.

Use transformation $x = au, y = bv, z = cw$, the upper half of the ellipsoid D becomes the upper half of the unit sphere D':

$$D' = \left\{(u,v,w) : u^2+v^2+w^2 \le 1,\, w \ge 0\right\}.$$

The Jacobian matrix is:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

The determinant of the Jacobian is:

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = abc.$$

The integral transforms as:

$$\iiint_D z^3\,dx\,dy\,dz = \iiint_{D'} (cw)^3 \cdot abc\,du\,dv\,dw = abc^4 \iiint_{D'} w^3\,du\,dv\,dw.$$

Over the Upper Half of the Unit Sphere, we use spherical coordinates:

$$u = r\sin\theta\cos\phi, \quad v = r\sin\theta\sin\phi, \quad w = r\cos\theta,$$

where: $0 \le r \le 1$, $0 \le \theta \le \pi/2$, $0 \le \phi \le 2\pi$.
The integral becomes:

$$abc^4 \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 r^5\cos^3\theta\sin\theta\,dr\,d\theta\,d\phi = \frac{abc^4\pi}{12}.$$

Problem 6: Find the circulation and flux of the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$ over the ellipse C given by $t \mapsto (\cos(t), 4\sin(t))$ for $t \in [0, 2\pi]$.

The dot product $\mathbf{F} \cdot \mathbf{r}'$ is:

$$\mathbf{F} \cdot \mathbf{r}' = \cos(t)(-\sin(t)) + 4\sin(t)(4\cos(t)) = -\cos(t)\sin(t) + 16\sin(t)\cos(t) = 15\sin(t)\cos(t).$$

The circulation is:

$$\text{Circulation} = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} 15\sin(t)\cos(t)\,dt = 0.$$

(Use the identity $\sin(2t) = 2\sin(t)\cos(t)$).
The flux of $\mathbf{F}$ across the curve C is given by the line integral:

$$\text{Flux} = \oint_C -N\,dx + M\,dy,$$

where $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$. For $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, we have $M = x$ and $N = y$. Thus,

$$\text{Flux} = \oint_C -y\,dx + x\,dy.$$

Parametrize the ellipse C as $\mathbf{r}(t) = (\cos(t), 4\sin(t))$ for $t \in [0, 2\pi]$. Then,

$$dx = -\sin(t)\,dt, \quad dy = 4\cos(t)\,dt.$$

Substitute $x = \cos(t)$, $y = 4\sin(t)$, $dx = -\sin(t)\,dt$, and $dy = 4\cos(t)\,dt$ into the flux integral:

$$\text{Flux} = \int_0^{2\pi} (-4\sin(t)(-\sin(t)) + \cos(t)(4\cos(t)))\,dt.$$

Simplify the integrand:

$$-4\sin(t)(-\sin(t)) + \cos(t)(4\cos(t)) = 4\sin^2(t) + 4\cos^2(t).$$

Using the identity $\sin^2(t) + \cos^2(t) = 1$, rewrite the integrand:

$$4\sin^2(t) + 4\cos^2(t) = 4(\sin^2(t) + \cos^2(t)) = 4.$$

Thus,

$$\text{Flux} = \int_0^{2\pi} 4\,dt = 8\pi.$$

Problem 7: Consider the vector field

$$\mathbf{F} = (e^x \ln(y))\mathbf{i} + (\frac{e^x}{y} + \sin(z))\mathbf{j} + (y\cos(z))\mathbf{k}.$$

defined over the region in $\mathbb{R}^3$ where $y > 0$. Determine whether $\mathbf{F}$ is conservative. If so, find a potential function. Otherwise, explain why.

The region with $y > 0$ on $\mathbb{R}^3$ is simply connected, so $\mathbf{F}$ is conservative if and only if it passes the component test:

$$\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = \cos(z) - \cos(z) = 0$$

$$\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = 0 - 0 = 0$$

$$\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{e^x}{y} - \frac{e^x}{y} = 0$$

Therefore, $\mathbf{F}$ is conservative. To find $f(x,y,z)$ such that $\nabla\phi = \mathbf{F}$, we first itegrate F_x with respect to x:

$$\phi(x,y,z) = e^x \ln(y) + g(y,z)$$

Then differentiate with respect to y and compare with F_y:

$$\frac{\partial \phi}{\partial y} = \frac{e^x}{y} + \frac{\partial g}{\partial y} = \frac{e^x}{y} + \sin(z)$$

$$\Rightarrow \frac{\partial g}{\partial y} = \sin(z) \Rightarrow g(y,z) = y\sin(z) + h(z)$$

Finally, differentiate with respect to z and compare with F_z:

$$\frac{\partial \phi}{\partial z} = y\cos(z) + h'(z) = y\cos(z)$$

$$\Rightarrow h'(z) = 0 \Rightarrow h(z) = C$$

Combining all terms:

$$f(x,y,z) = e^x \ln(y) + y\sin(z) + C$$

To receive full points for this question, there needs to be a clear indication that the region $y > 0$ in $\mathbb{R}^3$ is simply connected, or that the potential function is defined over exactly the same region. Sometimes a potential function exists locally but not globally and $\mathbf{F}$ *is still not conservative. However, if your answer to Problem 8 shows such concerns in mind, I deducted few points than otherwise.*

Problem 8: Let **F** be the vector field

$$\frac{-y}{x^2+y^2}\mathbf{i}+\frac{x}{x^2+y^2}\mathbf{j}.$$

defined on $\Omega = \mathbb{R}^2 \setminus \{(0,0)\}$. Determine whether **F** is conservative. If so, find a potential function. Otherwise, explain why.

Consider the unit circle C given by $\mathbf{r}(t) = (\cos(t), \sin(t))$ for $t \in [0, \pi]$. Then

$$\oint_C \mathbf{F} d\mathbf{r} = \int_0^{2\pi} (-\sin(t)\mathbf{i}+\cos(t)\mathbf{j}) \cdot (-\sin(t)\mathbf{i}+\cos(t)\mathbf{j}) dt = \int_0^{2\pi} 1 dt = 2\pi \neq 0.$$

Hence the vector field **F** is not conservative.